ON THE BIEBERBACH CONJECTURE FOR FUNCTIONS WITH A SMALL SECOND COEFFICIENT

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ABSTRACT

In the following we prove that for a given univalent function such that $|a_2| < 1.05$, $|a_n| < n$ for each n. This is an improvement of the result in [1].

1. Introduction

Let S denote the class of all normalized univalent functions in the unit disc U. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S$ then $|a_2| \leqq 2$. The Bieberbach conjecture states that $|a_n| \leqq n$ for every natural n. This conjecture is already known to be true for $n \leqq 6$. Many partial results are known in general. For instance it is known that if $f \in S$ then $|a_n| \leqq n$ provided $n > n_0(f)$ [5]. Als o if $f \in S$ then $|a_n| < 1.081n$[3] If $|a_2|$ is close enough to 2 the conjecture is known to be true [4]. On the other hand the conjecture is known to be true if $|a_2|$ is far from 2; more precisely, if $|a_2| < 0.867$ then $|a_n| < n$ [1]. It is the aim of this paper to improve the mentioned constant to 1.05.

2. Estimate for $|a_n|$ if $|a_2|$ is small

Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S$. Denote

$$\log \frac{f(z)}{z} = 2 \sum_{k=1}^{\infty} \gamma_k z^k. \tag{2.1}$$

We have

THEOREM A [1]. *If $f(z) \in S$ then $\sum_{k=1}^{n} k|\gamma_k|^2 < \sum_{k=1}^{n} 1/k + (|\gamma_1|^2 - \frac{1}{2} + \delta)$ where $\delta < 0.312$ is the Milin constant* [6].

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Let $\{A_k\}_{k=1}^{\infty}$ be an arbitrary sequence of complex numbers. Let $\{D_k\}_{k=0}^{\infty}$ be defined by $\exp(\sum_{k=1}^{\infty} A_k z^k) = \sum_{n=0}^{\infty} D_k z^k$. Then we have

THEOREM B [2], [7]. *The sequence* $\{P_n\}_{n=1}^{\infty}$ *defined by*

(i) $$P_n = \left(\frac{1}{n}\sum_{k=0}^{n-1} |D_k|^2\right)\exp\left[-\left(\sum_{k=1}^{n} k|A_k|^2 - \sum_{k=1}^{n}\frac{1}{k} + 1 - \frac{1}{n}\sum_{k=1}^{n} k^2|A_k|^2\right)\right]$$

satisfies $1 = P_1 \geqq P_2 \geqq \cdots \geqq P_n \geqq \cdots$. *Also*

(ii) $$|D_n|^2 \leqq P_l \exp\left(\sum_{k=1}^{n} k|A_k|^2 - \sum_{k=1}^{n}\frac{1}{k}\right), \qquad 1 \leqq l \leqq n.$$

Let $\sqrt{f(z^2)} = f_2(z) = \sum_{k=0}^{\infty} c_k z^{2k+1}$. We have $\log(f_2(z)/z) = \log\sqrt{f(z^2)/z^2} = \sum_{k=1}^{\infty}\gamma_k z^{2k}$.

Thus $f_2(z)/z = \sum_{k=0}^{\infty} c_k z^{2k} = \exp(\sum_{k=1}^{\infty}\gamma_k z^{2k})$ or

$$\exp\left(\sum_{k=1}^{\infty}\gamma_k z^k\right) = \sum_{k=0}^{\infty} c_k z^k. \tag{2.2}$$

Using Theorem B (ii) for $l = 2$ we have from (2.2)

$$|c_n|^2 \leqq \tfrac{1}{2}(1 + |c_1|^2)\exp\left[-\left(\frac{|\gamma_1|^2 - 1}{2}\right)\right]\exp\left(\sum_{k=1}^{n} k|\gamma_k|^2 - \sum_{k=1}^{n} 1/k\right). \tag{2.3}$$

But $c_1 = \gamma_1$. Thus we get from Theorem A and (2.3)

$$|c_n|^2 \leqq \tfrac{1}{2}\frac{1 + |\gamma_1|^2}{\exp[\frac{1}{2}(|\gamma_1|^2 - 1)]}\exp(|\gamma_1|^2 - \tfrac{1}{2} + \delta), \qquad \delta < 0.312. \tag{2.4}$$

We now have

THEOREM. *If* $|a_2| < 1.05$ *then* $|c_n| < 1$ *and* $|a_n| < n$ *for* $n \geqq 2$.

PROOF. $a_2/2 = \gamma_1$ and so $|\gamma_1| < 0.525$ implies with (2.4) and some elementary calculation that $|c_n| < 1$. But $f(z^2) = [f_2(z)]^2$ and so

$$a_n = \sum_{k=0}^{n-1} c_k c_{n-1-k} \tag{2.5}$$

or by the Schwarz inequality

$$|a_n| \leqq \sum_{k=0}^{n-1} |c_k|^2. \tag{2.6}$$

Thus $|c_n| < 1$ implies $|a_n| < n$.

REMARKS

1) We substituted $l = 2$ in (ii). Since P_l is monotone decreasing it is clear that for $l \geqq 3$ one must get better results, although the calculation may be more complicated.

2) Any improvement of the Milin constant δ will give an improvement of the above estimate.

3) It is worth noting that the remarkable proof of the Bieberbach conjecture for $n = 5$ in [8] has some connection to our work in the following sense. The authors first prove that if $|a_2|$ is close to 2 then $|a_5| < 5$. Then they extend the result globally to the whole range $|a_2| \leqq 2$.

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